THE FUTURE OF MATHEMATICS.[1]

TO foresee the future of mathematics, the true method is to study its history and its present state.

Is this not for us mathematicians in a way a professional procedure? We are accustomed to ***extrapolate,*** which is a means of deducing the future from the past and present, and as we well know what this amounts to, we run no risk of deceiving ourselves about the range of the results it gives us.

We have had hitherto prophets of evil. They blithely reiterate that all problems capable of solution have already been solved, and that nothing is left but gleaning. Happily the case of the past reassures us. Often it was thought all problems were solved, or at least an inventory was made of all admitting solution. And then the sense of the word solution enlarged, the insoluble problems became the most interesting of all, and others unforeseen presented themselves. For the Greeks a good solution was one employing only ruler and compasses; then it became one obtained by the extraction of roots, then one using only algebraic or logarithmic functions. The pessimists thus found themselves always outflanked, always forced to retreat, so that at present I think there are no more.

My intention therefore is not to combat them, as they are dead; we well know that mathematics will continue to develop, but the question is how, in what direction?

[1] Translated from the French by George Bruce Halsted.

You will answer, "in every direction," and that is partly true; but if it were wholly true it would be a little appalling. Our riches would soon become encumbering, and their accumulation would produce a medley as impenetrable as the unknown truth was for the ignorant.

The historian, the physicist even, must make a choice among facts; the head of the scientist, which is only a corner of the universe, could never contain the universe entire; so that among the innumerable facts nature offers, some will be passed by, others retained.

Just so, *a fortiori,* in mathematics; no more can the geometer hold fast pell-mell all the facts presenting themselves to him; all the more because he it is, almost I had said his caprice, that creates these facts. He constructs a wholly new combination by putting together its elements; nature does not in general give it to him ready made.

Doubtless it sometimes happens that the mathematician undertakes a problem to satisfy a need in physics; that the physicist or engineer asks him to calculate a number for a certain application. Shall it be said that we geometers should limit ourselves to awaiting orders, and, in place of cultivating our science for our own delectation, try only to accommodate ourselves to the wants of our patrons? If mathematics has no other object besides aiding those who study nature, it is from these we should await orders. Is this way of looking at it legitimate? Certainly not; if we had not cultivated the exact sciences for themselves, we would not have created mathematics the instrument, and the day the call came from the physicist, we would have been helpless.

Nor do the physicists wait to study a phenomenon until some urgent need of material life has made it a necessity for them; and they are right. If the scientists of the eighteenth century had neglected electricity as being in their eyes only a curiosity without practical interest, we

should have had in the twentieth century neither telegraphy, nor electro-chemistry, nor electro-technics. The physicists, compelled to choose, are therefore not guided in their choice solely by utility. How then do they choose between the facts of nature? We have explained it in a previous article:[2] the facts which interest them are those capable of leading to the discovery of a law, and so they are analogous to many other facts which do not seem to us isolated, but closely grouped with others. The isolated fact attracts all eyes, those of the layman as well as of the scientist. But what the genuine physicist alone knows how to see, is the bond which unites many facts whose analogy is profound but hidden. The story of Newton's apple is probably not true, but it is symbolic; let us speak of it then as if it were true. Well then, we must believe that before Newton plenty of men had seen apples fall; not one knew how to conclude anything therefrom. Facts would be sterile were there not minds capable of choosing among them, discerning those behind which something was hidden, and of recognizing what is hiding, minds which under the crude fact perceive the soul of the fact.

We find just the same thing in mathematics. From the varied elements at our disposal we can get millions of different combinations; but one of these combinations, in so far as it is isolated, is absolutely void of value. Often we have taken great pains to construct it, but it serves no purpose, if not perhaps to furnish a task in secondary education. Quite otherwise will it be when this combination shall find place in a class of analogous combinations and we shall have noticed this analogy. We are no longer in the presence of a fact but of a law. And upon that day the real discoverer will not be the workman who shall have patiently built up certain of these combinations; it will be he who brings to light their kinship. The first will

[2] See "The Choice of Facts," *The Monist*, April, 1909.

have seen merely the crude fact, only the other will have perceived the soul of the fact. Often to fix this kinship it suffices him to make a new word, and this word is creative. The history of science furnishes us a crowd of examples familiar to all.

The celebrated Vienna philosopher Mach has said that the rôle of science is to produce economy of thought, just as machines produce economy of effort. And that is very true. The savage reckons on his fingers or by heaping pebbles. In teaching children the multiplication table we spare them later innumerable pebble bunchings. Some one has already found out with pebbles or otherwise, that 6 times 7 is 42 and has had the idea of noting the result, and so we need not do it over again. He did not waste his time even if he reckoned for pleasure: his operation took him only two minutes; it would have taken in all two milliards if a milliard men had had to do it over after him.

The importance of a fact then is measured by its yield, that is to say, by the amount of thought it permits us to spare.

In physics the facts of great yield are those entering into a very general law, since from it they enable us to foresee a great number of others, and just so it is in mathematics. Suppose I have undertaken a complicated calculation and laboriously reached a result: I shall not be compensated for my trouble if thereby I have not become capable of foreseeing the results of other analogous calculations and guiding them with a certainty that avoids the gropings to which one must be resigned in a first attempt. On the other hand I shall not have wasted my time if these gropings themselves have ended by revealing to me the profound analogy of the problem just treated with a much more extended class of other problems; if they have shown me at once the resemblances and differences of these, if in a word they have made me perceive the possibility of a

generalization. Then it is not a new result I have won, it is a new power.

The simple example that comes first to mind is that of an algebraic formula which gives us the solution of a type of numeric problems when finally we replace the letters by numbers. Thanks to it a single algebraic calculation saves us the pains of ceaselessly beginning over again new numeric calculations. But this is only a crude example; we all know there are analogies inexpressible by a formula and all the more precious.

A new result is of value, if at all, when in unifying elements long known but hitherto separate and seeming strangers one to another, it suddenly introduces order where apparently disorder reigned. It then permits us to see at a glance each of these elements and its place in the assemblage. This new fact is not merely precious by itself, but it alone gives value to all the old facts it combines. Our mind is weak as are the senses; it would lose itself in the world's complexity were this complexity not harmonious; like a near-sighted person it would see only the details and would be forced to forget each of these details before examining the following, since it would be incapable of embracing all. The only facts worthy our attention are those which introduce order into this complexity and so make it accessible.

Mathematicians attach great importance to the elegance of their methods and their results. This is not pure dilettantism. What is it indeed that gives us the feeling of elegance in a solution, in a demonstration? It is the harmony of the diverse parts, their symmetry, their happy balance; in a word it is all that introduces order, all that gives unity, that permits us to see clearly and to comprehend at once both the *ensemble* and the details. But this is exactly what yields great results; in fact the more we see this aggregate clearly and at a single glance, the better

we perceive its analogies with other neighboring objects, consequently the more chances we have of divining the possible generalizations. Elegance may produce the feeling of the unforeseen by the unexpected meeting of objects we are not accustomed to bring together; there again it is fruitful, since it thus unveils for us kinships before unrecognized. It is fruitful even when it results only from the contrast between the simplicity of the means and the complexity of the problem set; it makes us then think of the reason for this contrast and very often makes us see that chance is not the reason; that it is to be found in some unexpected law. In a word, the feeling of mathematical elegance is only the satisfaction due to any adaptation of the solution to the needs of our mind, and it is because of this very adaptation that this solution can be for us an instrument. Consequently this esthetic satisfaction is bound up with the economy of thought. Again the comparison of the Erechtheum comes to my mind, but I must not use it too often.

It is for the same reason that, when a rather long calculation has led to some simple and striking result, we are not satisfied until we have shown that we should have been *able to foresee,* if not this entire result, at least its most characteristic traits. Why? What prevents our being content with a calculation which has told us, it seems, all we wished to know? It is because, in analogous cases, the long calculation might not again avail, and that this is not so about the reasoning often half intuitive which would have enabled us to foresee. This reasoning being short, we see at a single glance all its parts, so that we immediately perceive what must be changed to adapt it to all the problems of the same nature which can occur. And then it enables us to foresee if the solution of these problems will be simple, it shows us at least if the calculation is worth undertaking.

What we have just said suffices to show how vain it would be to seek to replace by any mechanical procedure the free initiative of the mathematician. To obtain a result of real value, it is not enough to grind out calculations, or to have a machine to put things in order; it is not order alone, it is unexpected order which is worth while. The machine may gnaw on the crude fact, the soul of the fact will always escape it.

Since the middle of the last century, mathematicians are more and more desirous of attaining absolute rigor; they are right, and this tendency will be more and more accentuated. In mathematics rigor is not everything, but without it there is nothing. A demonstration which is not rigorous is nothingness. I think no one will contest this truth. But if it were taken too literally, we should be led to conclude that before 1820, for example, there was no mathematics; this would be manifestly excessive; the geometers of that time understood voluntarily what we explain by prolix discourse. This does not mean that they did not see it at all; but they passed over too rapidly, and to see it well would have necessitated taking the pains to say it.

But is it always needful to say it so many times; those who were the first to emphasize exactness before all else have given us arguments that we may try to imitate; but if the demonstrations of the future are to be built on this model, mathematical treatises will be very long; and if I fear the lengthenings, it is not solely because I deprecate encumbering libraries, but because I fear that in being lengthened out, our demonstrations may lose that appearance of harmony whose usefulness I have just explained.

The economy of thought is what we should aim at, so it is not enough to supply models for imitation. It is needful for those after us to be able to dispense with these models and in place of repeating an argument already

made, summarize it in a few words. And this has already been attained at times. For instance there was a type of reasoning found everywhere, and everywhere alike. They were perfectly exact, but long. Then all at once the phrase "uniformity of convergence" was hit upon, and this phrase made those arguments needless; we were no longer called upon to repeat them, since they could be understood. Those who conquer difficulties then do us a double service; first they teach us to do as they at need, but above all they enable us as often as possible to avoid doing as they, yet without sacrifice of exactness.

We have just seen by one example the importance of words in mathematics, but many others could be cited. It is hard to believe how much a well-chosen word can economize thought, as Mach says. Perhaps I have already said somewhere that mathematics is the art of giving the same name to different things. It is fitting that these things, differing in matter, may be alike in form, that they may, so to speak, run in the same mould. When the language has been well chosen, we are astonished to see that all the proofs made for a certain object apply immediately to many new objects; there is nothing to change, not even the words, since the names have become the same.

A well-chosen word usually suffices to do away with the exceptions from which the rules stated in the old way suffer; this is why we have created negative quantities, imaginaries, points at infinity, and what not. And exceptions, we must not forget, are pernicious because they hide the laws.

Well, this is one of the characteristics by which we recognize the facts which yield great results. They are those which allow of these happy innovations of language. The crude fact then is often of no great interest; we may point it out many times without having rendered great service to science. It takes value only when a wiser thinker per-

ceives the relation for which it stands, and symbolizes it by a word.

Moreover the physicists do just the same. They have invented the word "energy," and this word has been prodigiously fruitful, because it also made the law by eliminating the exceptions, since it gave the same name to things differing in matter and like in form.

Among words that have had the most fortunate influence I would select "group" and "invariant." They have made us see the essence of many mathematical reasonings; they have shown us in how many cases the old mathematicians considered groups without knowing it, and how, believing themselves far from one another, they suddenly found themselves near without knowing why.

To-day we should say that they had dealt with isomorphic groups. We now know that in a group the matter is of little interest, the form alone counts, and that when we know a group we thus know all the isomorphic groups; and thanks to these words "group" and "isomorphism," which condense in a few syllables this subtile rule and quickly make it familiar to all minds, the transition is immediate and can be done with every economy of thought effort. The idea of group besides attaches to that of transformation. Why do we put such a value on the invention of a new transformation? Because from a single theorem it enables us to get ten or twenty; it has the same value as a zero adjoined to the right of a whole number.

This then it is which has hitherto determined the direction of mathematical advance, and just as certainly will determine it in the future. But to this end the nature of the problems which come up contributes equally. We can not forget what must be our aim. In my opinion this aim is double. Our science borders upon both philosophy and physics, and we work for our two neighbors; so we have

always seen and shall still see mathematicians advancing in two opposite directions.

On the one hand, mathematical science must reflect upon itself, and that is useful since reflecting on itself is reflecting on the human mind which has created it, all the more because it is the very one of its creations for which it has borrowed least from without. This is why certain mathematical speculations are useful, such as those devoted to the study of the postulates, of unusual geometries, of peculiar functions. The more these speculations diverge from ordinary conceptions, and consequently from nature and applications, the better they show us what the human mind can create when it frees itself more and more from the tyranny of the external world, so the better they let us know it in itself.

But it is toward the other side, the side of nature that we must direct the bulk of our army. There we meet the physicist or the engineer, who says to us: "Please integrate this differential equation for me; I might need it in a week in view of a construction which should be finished by that time." "This equation," we answer, "does not come under one of the integrable types; you know there are not many." "Yes, I know; but then what good are you?" Usually to understand each other is enough; the engineer in reality does not need the integral in finite terms; he needs to know the general look of the integral function, or he simply wants a certain number which could readily be deduced from this integral if it were known. Usually it is not known, but the number can be calculated without it if we know exactly what number the engineer needs and with what approximation.

Formerly an equation was considered solved only when its solution had been expressed by aid of a finite number of known functions; but that is possible scarcely once in a hundred times. What we always can do, or rather what

we should always seek to do, is to solve the problem *qualitatively* so to speak; that is to say, seek to know the general form of the curve which represents the unknown function.

It remains to find the *quantitative* solution of the problem; but if the unknown cannot be determined by a finite calculation, it may always be represented by a convergent infinite series which enables us to calculate it. Can that be regarded as a true solution? We are told that Newton sent Leibnitz an anagram almost like this: *aaaaabbbeeeeii,* etc. Leibnitz naturally understood nothing at all of it; but we, who have the key, know that this anagram meant translated into modern terms: "I can integrate all differential equations"; and we are tempted to say that Newton had either great luck or strange delusions. He merely wished to say he could form (by the method of indeterminate coefficients) a series of powers formally satisfying the proposed equation.

Such a solution would not satisfy us to-day, and for two reasons: because the convergence is too slow and because the terms follow each other without obeying any law. On the contrary, the series Θ seems to us to leave nothing to be desired, first because it converges very quickly (this is for the practical man who wishes to get at a number as quickly as possible) and next because we see at a glance the law of the terms (this is to satisfy the esthetic need of the theorist).

But then there are no longer solved problems and others which are not; there are only problems *more or less* solved according as they are solved by a series converging more or less rapidly, or ruled by a law more or less harmonious. It often happens however that an imperfect solution guides us toward a better one. Sometimes the series converges so slowly that the computation is impracticable and we have only succeeded in proving the possibility of the problem.

And then the engineer finds this a mockery, and justly, since it will not aid him to complete his construction by the date fixed. He little cares to know if it will benefit engineers of the twenty-second century. But as for us, we think differently and we are sometimes happier to have spared our grand-children a day's work than to have saved our contemporaries an hour.

Sometimes by groping, empirically so to speak, we reach a formula sufficiently convergent. "What more do do you want?" says the engineer. And yet, in spite of all, we are not satisfied; we should have liked *to foresee* that convergence. Why? Because if we had known how to foresee it once, we would know how to foresee it another time. We have succeeded; that is a small matter in our eyes if we cannot validly expect to do so again.

In proportion as science develops its total, comprehension becomes more difficult; then we seek to cut it in pieces and to be satisfied with one of these pieces: in a word, to specialize. If we went on in this way, it would be a grievous obstacle to the progress of science. As we have said, it is by unexpected unions between its diverse parts that it progresses. To specialize too much would be to forbid these drawings together. It is to be hoped that congresses like those of Heidelberg and Rome, by putting us in touch with one another will open for us vistas over neighboring domains and oblige us to compare them with our own, to range somewhat abroad from our own little village; thus they will be the best remedy for the danger just mentioned.

But I have lingered too long over generalities, it is time to enter into detail.

Let us pass in review the various special sciences which combined make mathematics; let us see what each has accomplished, whither it tends and what we may hope from it. If the preceding views are correct, we should see

that the greatest advances in the past have happened when two of these sciences have united, when we have become conscious of the similarity of their form, despite the difference of their matter, when they have so modeled themselves upon each other that each could profit by the other's conquests. We should at the same time foresee in combinations of the same sort, the progress of the future.

ARITHMETIC.

Progress in arithmetic has been much slower than in algebra and analysis, and it is easy to see why. The feeling of continuity is a precious guide which the arithmetician lacks; each whole number is separated from the others, —it has, so to speak, its own individuality. Each of them is a sort of exception and this is why general theorems are rarer in the theory of numbers; this is also why those which exist are more hidden and longer elude the searchers.

If arithmetic is behind algebra and analysis, the best thing for it to do is to seek to model itself upon these sciences so as to profit by their advance. The arithmetician ought therefore to take as guide the analogies with algebra. These analogies are numerous and if, in many cases, they have not yet been studied sufficiently closely to become utilizable, they at least have long been foreseen, and even the language of the two sciences shows they have been recognized. Thus we speak of transcendent numbers and thus we account for the future classification of these numbers already having as model the classification of transcendent functions, and still we do not as yet very well see how to pass from one classification to the other; but had it been seen, it would already have been accomplished and would no longer be the work of the future.

The first example that comes to my mind is the theory of congruences where is found a perfect parallelism to the

theory of algebraic equations. Surely we shall succeed in completing this parallelism, which must hold for instance between the theory of algebraic curves and that of congruences with two variables. And when the problems relative to congruences with several variables shall be solved, this will be a first step toward the solution of many questions of indeterminate analysis.

ALGEBRA.

The theory of algebraic equations will still long hold the attention of geometers; numerous and very different are the sides whence it may be attacked.

We need not think algebra is ended because it gives us rules to form all possible combinations; it remains to find the interesting combinations, those which satisfy such and such a condition. Thus will be formed a sort of indeterminate analysis where the unknowns will no longer be whole numbers, but polynomials. This time it is algebra which will model itself upon arithmetic, following the analogy of the whole number to the integral polynomial with any coefficients or to the integral polynomial with integral coefficients.

GEOMETRY.

It looks as if geometry could contain nothing which is not already included in algebra or analysis; that geometric facts are only algebraic or analytic facts expressed in another language. It might then be thought that after our review there would remain nothing more for us to say relating specially to geometry. This would be to fail to recognize the importance of well constructed language, not to comprehend what is added to the things themselves by the method of expressing these things and consequently of grouping them.

First the geometric considerations lead us to set our-

selves new problems; these may be, if you choose, analytic problems, but such as we never would have set ourselves in connection with analysis. Analysis profits by them however, as it profits by those it has to solve to satisfy the needs of physics.

A great advantage of geometry lies in the fact that in it the senses can come to the aid of thought, and help find the path to follow, and many minds prefer to put the problems of analysis into geometric form. Unhappily our senses cannot carry us very far, and they desert us when we wish to soar beyond the classical three dimensions. Does this mean that, beyond the restricted domain wherein they seem to wish to imprison us, we should rely only on pure analysis and that all geometry of more than three dimensions is vain and objectless? The greatest masters of a preceding generation would have answered "yes"; to-day we are so familiarized with this notion that we can speak of it, even in a university course, without arousing too much astonishment.

But what good is it? That is easy to see: First it gives us a very convenient terminology, which expresses concisely what the ordinary analytic language would say in prolix phrases. Moreover, this language makes us call like things by the same name and emphasize analogies it will never again let us forget. It enables us therefore still to find our way in this space which is too big for us and which we cannot see, always recalling visible space which is only an imperfect image of it doubtless, but which is nevertheless an image. Here again, as in all the preceding examples, it is analogy with the simple which enables us to comprehend the complex.

This geometry of more than three dimensions is not a simple analytic geometry; it is not purely quantitative, but qualitative also, and it is in this respect above all that it becomes interesting. There is a science called *Analysis*

Situs and which has for its object the study of the positional relations of the different elements of a figure, apart from their sizes. This geometry is purely qualitative; its theorems would remain true if the figures, instead of being exact, were roughly imitated by a child. We may also make an *Analysis Situs* of more than three dimensions. The importance of *Analysis Situs* is enormous and cannot be too much emphasized; the advantage obtained from it by Riemann, one of its chief creators, would suffice to prove this. We must achieve its complete construction in the higher spaces; then we shall have an instrument which will enable us really to see in hyperspace and supplement our senses.

The problems of *Analysis Situs* would perhaps not have suggested themselves if the analytic language alone had been spoken; or rather, I am mistaken, they would have occurred surely, since their solution is essential to a crowd of questions in analysis, but they would have come singly, one after another, and without our being able to perceive their common bond.

CANTORISM.

I have spoken above of our need to go back continually to the first principles of our science, and of the advantage of this for the study of the human mind. This need has inspired two endeavors which have taken a very prominent place in the most recent annals of mathematics. The first is Cantorism which has rendered our science such conspicuous service. Cantor introduced into science a new way of considering mathematical infinity. One of the characteristic traits of Cantorism is that in place of going up to the general by building up constructions more and more complicated and defining by construction, it starts from the *genus supremum* and defines only, as the scholastics would have said, *per genus proximum et differentiam spe-*

cificam. Thence comes the horror it has sometimes inspired in certain minds, for instance in Hermite, whose favorite idea was to compare the mathematical to the natural sciences. With most of us these prejudices have been dissipated, but it has come to pass that we have encountered certain paradoxes, certain apparent contradictions that would have delighted Zeno the Eleatic and the school of Megara. And then each must seek the remedy. For my part, I think, and I am not the only one, that the important thing is never to introduce entities not completely definable in a finite number of words. Whatever be the cure adopted, we may promise ourselves the joy of the doctor called in to follow a beautiful pathologic case.

THE INVESTIGATION OF THE POSTULATES.

On the other hand, efforts have been made to enumerate the axioms and postulates, more or less hidden, which serve as foundation to the different theories of mathematics. Professor Hilbert has obtained the most brilliant results. It seems at first that this domain would be very restricted and there would be nothing more to do when the inventory should be ended, which could not take long. But when we shall have enumerated all, there will be many ways of classifying all; a good librarian always finds something to do, and each new classification will be instructive for the philosopher.

Here I end this review which I could not dream of making complete. I think these examples will suffice to show by what mechanism the mathematical sciences have made their progress in the past and in what direction they must advance in the future.

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